Fractal geometry of critical systems

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We investigate the geometry of a critical system undergoing a second-order thermal phase transition. Using a local description for the dynamics characterizing the system at the critical point $T=T_c$, we reveal the formation of clusters with fractal geometry, where the term cluster is used to describe regions with a nonvanishing value of the order parameter. We show that, treating the cluster as an open subsystem of the entire system, new instanton-like configurations dominate the statistical mechanics of the cluster. We study the dependence of the resulting fractal dimension on the embedding dimension and the scaling properties (isothermal critical exponent) of the system. Taking into account the finite-size effects, we are able to calculate the size of the critical cluster in terms of the total size of the system, the critical temperature, and the effective coupling of the long wavelength interaction at the critical point. We also show that the size of the cluster has to be identified with the correlation length at criticality. Finally, within the framework of the mean field approximation, we extend our local considerations to obtain a global description of the system.

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I. INTRODUCTION

Understanding the geometry of systems near a second-order critical point is the subject of numerous recent works [1]. Most of these works considered dynamics in discrete space (lattice), and tried to explain the formation of clusters with fractal geometry on the embedding lattice in terms of the scaling properties (critical exponents) of the system [2,3]. In a recent work [4], we proposed a mechanism in order to understand the formation of fractal clusters at $T=T_c$ for systems defined in a continuous space. Based on a scale invariant effective action describing the dynamics at the critical point, we were mainly interested in revealing how this dynamics leads to the formation of critical clusters. A general class of saddle points of the effective action at $T=T_c$ turns out to dominate the configurations contributing to the partition function if we consider the statistical mechanics of an open subsystem (cluster) of the global critical system. In the present work we present in more detail and completeness the method used in Ref. [4] to obtain a consistent picture of the local geometry at the transition point for one-dimensional systems, and we then apply our approach in order to describe critical systems in higher dimensions. We also take into account finite-size effects, and we discuss the possibility of using different functional realizations for the order parameter characterizing the system at the critical point. Based on a local description of the critical system we propose an algorithm, using arguments within the framework of the mean field approximation, to construct a global system and to determine its scaling properties.

The starting point in our investigation is the effective action of a thermal system at the critical point $T=T_c$, specified in d dimensions in terms of a macroscopic field ϕ (order parameter) as follows:

$$\Gamma_c[\phi]=g_1\Lambda^{\beta}\int d^dx\left[\frac{1}{2}(\nabla_d\phi)^2+g_2\Lambda^{\gamma}|\Lambda^{-d}\phi|^{\delta+1}\right]. \tag{1}$$

In Eq. (1), g_1 and g_2 are dimensionless couplings, $\phi \sim$ (volume)$^{-1}$, and the ultraviolet cutoff Λ of the underlying microscopic theory fixes the coarse graining scale $R_c \approx \Lambda^{-1}$ of the effective system. Equation (1) leads to the standard equation of state at $T=T_c$,

$$\frac{\delta\Gamma_c}{\delta\phi}\sim\phi^{\delta} \quad (\phi>0),$$

and the index δ is identified with the isothermal critical exponent of the system. Action (1), being dimensionless, implies $\beta=-(d+2)$ and $\gamma=2d+2$. Also introducing the dimensionless quantities $\hat{\phi}=\Lambda^{-d}\phi$ and $\hat{x}_i=\Lambda x_i$, we rewrite the effective action (1) as follows:

$$\Gamma_c[\hat{\phi}]=g_1\int d^d\hat{x}\left[\frac{1}{2}(\nabla\hat{\phi})^2+g_2|\hat{\phi}|^{\delta+1}\right]. \tag{2}$$

The scalar quantity ϕ describes in general the density of an extensive physical quantity characterizing the phase transition (like, for example, magnetization density or particle density). Let us now mention some examples of theories which belong to the class of physical systems described through the effective action [Eq. (2)].

$O(N)$ *three-dimensional (3D) effective theory*: The action, in the large N limit, for spherically symmetric order parameter in the internal $O(N)$ space, is written as [5]

$$\Gamma_c[\phi]=\lambda^5\Lambda^{-5}\int d^3\vec{x}\left[\frac{1}{2}(\nabla\phi)^2+\frac{2}{3}\left(\frac{2\pi\lambda^5}{N}\right)^2\Lambda^8(\Lambda^{-3}\phi)^6\right],$$

where $\lambda=\Lambda/T_c$. This action, for $\hat{\phi}=\Lambda^{-3}\phi$ and $\hat{\vec{x}}=\Lambda\vec{x}$, has the form

$$\Gamma_c[\hat{\phi}]=\lambda^5\int d^3\hat{\vec{x}}\left[\frac{1}{2}(\nabla\hat{\phi})^2+\frac{2}{3}\left(\frac{2\pi\lambda^5}{N}\right)^2|\hat{\phi}|^6\right].$$

It belongs to the general class of Eq. (2), with $g_1=\lambda^5$, $g_2=\frac{2}{3}(2\pi\lambda^5/N)^2$, $d=3$, and $\delta=5$.

3D Ising model: The effective action $\Gamma_c[\sigma]$ which effectively describes the QCD at the critical point ($T=T_c$) [6], is written as

$$\Gamma_c[\sigma]=T_c^{-1}\int d^3\vec{x}\left[\frac{1}{2}(\nabla\sigma)^2+GT_c^4(T_c^{-1}\sigma)^{\delta+1}\right],$$

where the macroscopic field $\sigma\sim$ (length)$^{-1}$. This action, for $\hat{\sigma}=\Lambda^{-1}\sigma$ and $\hat{\vec{x}}=\vec{x}\Lambda$, has the form

$$\Gamma_c[\hat{\sigma}]=\lambda\int d^3\hat{\vec{x}}\left[\frac{1}{2}(\nabla\hat{\sigma})^2+G\lambda^{\delta-3}\hat{\sigma}^{\delta+1}\right].$$

It belongs to the general class of Eq. (2), with $g_1=\lambda$, $g_2=G\lambda^{\delta-3}$, and $d=3$. We recall that $\lambda=\Lambda/T_c$.

Throughout this work we use the convention $\kappa_B=1$ (Boltzmann constant), and the energy is given in inverse length units. We will also use, for simplicity, the notation (ϕ,x_i) instead of $(\hat{\phi},\hat{x}_i)$ meaning always, unless otherwise stated, dimensionless quantities.

The paper is organized as follows: In Sec. II we investigate the statistical mechanics of the critical system for $d=1$ (d is the embedding space dimension) described by an effective action of the form of Eq. (2). The formation of fractal clusters is shown, and the corresponding geometrical characteristics (size and dimension) are determined. In Sec. III we extend the analysis to higher dimensions. In Sec. IV we study the dependence of the geometrical properties of the critical clusters on the coarse graining scale of the effective theory. We also apply our approach to critical systems with a more general functional form of the order parameter. Taking care of the finite-size effects, we determine the correlation length in terms of the size of the formed clusters. Then using a mean field approach, we construct the global system as a superposition of individual clusters and we explore its scaling (geometric) properties. Finally in Sec. V we summarize our main results and give a brief outlook. Some lengthy formulas referred to in the main text are given in the Appendix.

II. 1D SYSTEM

The statistical mechanics of the critical system is determined through the partition function

$$Z_1=\int \mathcal{D}[\phi]e^{-\Gamma_c[\phi]}. \tag{3}$$

The local description implies that the integration measure in Eq. (3) is over field configurations defined in an open ball Ω, with radius R and center $\overline{x}$, a subset of the space V (which in fact can be infinite) occupied by the entire system. We call a cluster C the set consisting of points belonging to Ω ($C\subset\Omega$), for which the order parameter ϕ is greater than or equal to a minimum value (cutoff) ϕ_{min}. We then identify $\overline{x}$ with the center of the cluster C. Without loss of generality we can set $\overline{x}=0$. The *local* geometrical properties of the system are determined through the scaling properties of the extensive quantities characterizing the cluster C as we vary the radius R. In order to illustrate our method we will first consider, for simplicity, the one-dimensional case. However, the extension to higher dimensions, as we will see in Sec. III, is straightforward.

In the one-dimensional case the effective action of the critical system ($T=T_c$), resulting from Eq. (2), is given by

$$\Gamma_c[\phi]=g_1\int_0^R dx\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+g_2|\phi|^{\delta+1}\right]. \tag{4}$$

Here we will consider models for which the condition $g_1\gg1$ is valid. This requirement allows us to use the saddle-point approximation to evaluate the path integration in Eq. (3) by replacing it through a summation over the saddle points of action (4). As the subsystem Ω is *open*, no boundary conditions restrict the configurations which contribute to the path integral (3). This point of view is essential in our approach.

The saddle-point configurations $\phi(x)$ fulfill the Euler-Lagrange equation $d^2\phi/dx^2=-\partial U(\phi)/\partial\phi$, where $U(\phi)$ is the concave pontential $U(\phi)=-g_2|\phi|^{\delta+1}$. Considering this equation as the motion of a classical particle, we obtain the first-order equation

$$E=\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2-g_2|\phi|^{\delta+1}, \tag{5}$$

where E is a conserved (during the classical motion) quantity identified with the total energy of the moving particle. Equation (5) can be integrated (for details, see the Appendix) to give, for $E=0$, instanton-like solutions of the form

$$\phi(x)=A_1|x-x_o|^{-2/(\delta-1)},\quad A_1=\left[\frac{g_2}{2}(\delta-1)^2\right]^{-1/(\delta-1)}, \tag{6}$$

with

$$x_o=\sqrt{2}/(\delta-1)\sqrt{g_2}(\phi(0))^{-2/(\delta-1)}.$$

Thus, for $E=0$, the position of the singularity x_o depends only on the initial condition $\phi(0)$. For $E\neq0$ the solution has the form $\phi(x)=A_1|x-x_o'|^{-2/(\delta-1)}$, where now $x_o'=x_o'(\phi(0),E)$ (see the Appendix). However, configurations with $E\neq0$, contribute to the partition function Z with a suppression factor $e^{-g_1R|E|}$, suggesting that the dominant saddle points in the path summation [Eq. (3)] are those solutions of the equation of motion for which $E\approx0$. In this case, Eq. (4) simplifies to

$$\Gamma_c[\phi]=2g_1g_2\int_0^R dx(\phi(x))^{\delta+1}.$$

Only configurations with $x_o>R$ give a nonvanishing contribution to the path integral [Eq. (3)]. In fact, the partition function is dominated by those saddle points for which $x_o\gg R$. Since $x\in(-R,R)$ we can easily take $\phi(x)=\text{const}=A_1x_o^{-2/(\delta-1)}$. It is straightforward to show that these solutions correspond to the long wavelength modes of the field $\phi(x)$ by taking the Fourier transform of Eq. (6). We obtain $f(k)\sim e^{ikx_o}/[(\delta-1)k^{\delta-3}]$, and the envelope of $f(k)$ is given by $k\sim m/x_o$.

Using the above approximation, we have

$$\Gamma_c = G_1 R x_o^{-2[(\delta+1)/(\delta-1)]}, \tag{7}$$

with $G_1=2g_1g_2A_1^{\delta+1}$. The summation over the saddle points of action (4) becomes, within this approximation, an ordinary integration over x_o with measure: $\mathcal{D}\phi = d\mu(x_o) \approx x_o^{-[(\delta+1)/(\delta-1)]}dx_o$. As stated above the singularity x_o must be located outside the cluster C to give a nonvanishing contribution to the partition function of the one-dimensional system. This condition fixes the lower limit in the integration over x_o to be $x_{o,min}=R$. To determine the upper limit of x_o one has to go back to the definition of the cluster C given previously. Without loss of generality, we can take the extensive quantity characterizing the geometry in C to be the magnetization $M=\int_0^R \phi(x)dx$ fulfilling the condition $M \geqslant \mu$, with $\mu = \int_0^R \phi_{min} dx = R\phi_{min}$. In the approximation of constant configurations for the order parameter ϕ we obtain the upper limit for x_o as $x_o \leqslant (AR/\mu)^{(\delta-1)/2}$.

The one-dimensional partition function in now written as

$$Z_1=\int_R^{(A_1R/\mu)^{(\delta-1)/2}} dx_o x_o^{-[(\delta+1)/(\delta-1)]} e^{-G_1 R x_o^{-2[(\delta+1)/(\delta-1)]}}.$$

Using this expression, it is straightforward to calculate the mean value of the magnetization in the cluster C:

$$\left\langle \int_0^R \phi(x)dx \right\rangle = \frac{A_1}{Z}\int_R^{(A_1R/\mu)^{(\delta-1)/2}} dx_o x_o^{-[(\delta+1)/(\delta-1)} \times \left(\int_0^R dx A_1 x_o^{-2/(\delta-1)} \right) e^{-G_1 R x_o^{-2[(\delta+1)/(\delta-1)]}}. \tag{8}$$

Using Eq. (8), we can show analytically (see Sec. IV) that in the large G_1 limit ($G_1 \gg 1$) there are three characteristic regions determining the behavior of the integral in Eq. (8). Setting $R_d = A_1^{-(\delta+1)/\delta}\mu^{(\delta+1)/\delta}G_1^{1/\delta}$ and $R_u = G_1^{(\delta-1)/(\delta+1)}$, we find the following

(i) For the region $R_d \ll R \ll R_u$, we have $\langle \int_0^R \phi(x)dx \rangle \sim R^{\delta/(\delta+1)}$, with coefficient

$$\alpha_1 \approx A_1 G_1^{-1/(\delta+1)} \frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)},$$

leading to a fractal structure of the cluster around the point $x=0$ with fractal mass dimension [7,8] $d_F=\delta/(\delta+1)$.

(ii) This behavior crosses over for $R \gg R_u$ to a different power law, $<\int_0^R \phi(x)dx> \sim R^{(\delta-3)/(\delta-1)}$ suggesting the presence of a fractal with mass dimension $\tilde{d}_F = (\delta-3)/(\delta-1)$ at large scales.

(iii) The lower limit R_d defines a minimal scale of the critical system, below which the fractality is broken.

The fractality in the central region characterizes the critical system in the sense that it corresponds to the scaling behavior in the vicinity of the local observer when $\mu \to 0$. The crossover scale R_u gives a measure of the correlation length of the finite system at $T=T_c$. In Fig. 1(a) we show the numerical results for the calculation of $\langle M \rangle$ using the

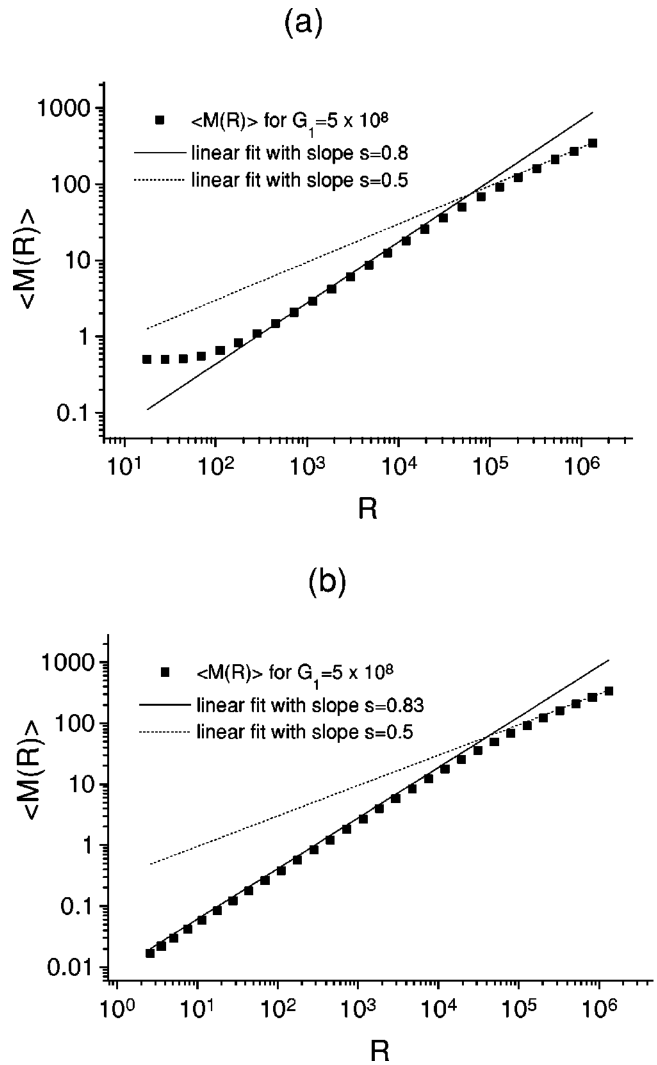

FIG. 1. (a) The mean magnetization $\langle M(R) \rangle$ as a function of R for a 1D critical system. The parameters are chosen so that $G_1=5\times10^8$ and $\mu=1$. A linear fit is also shown in order to indicate the two different fractality regions described in Sec. II. (b) The same plot as in (a), but now with $\mu=0$. The scale R_d for the breaking of the fractality in this case is absent. All presented quantities are in arbitrary units.

values $G_1=5\times10^8$ and $\delta=5$. We recognize the central region and the two scales R_d and R_u. In Fig. 1(b) we plot, for the same values of G_1 and δ, the results for the mean magnetization if we ignore the breaking at $R_d(\mu\to0)$.

III. FRACTAL CLUSTERS FOR $D>1$

Let us now extend our approach to higher dimensions starting from the two-dimensional case. We write the effective action (2) for $d=2$,

$$\Gamma_c[\phi] = g_1 \int d^2\vec{r} \left[\frac{1}{2}|\nabla\phi|^2 + g_2|\phi|^{\delta+1} \right],$$

and look for classical saddle points in an open subset of $\mathcal{R}^2$. The Euler-Lagrange equation, in this case, has the form

$$\nabla^2\phi = (\delta+1)g_2\phi^\delta,$$

and the corresponding instanton-like saddle points are:

$$d=2, \quad \phi_2(\vec{r})=A_2|\vec{r}-\vec{r}_o|^{-2/(\delta-1)},$$
$$A_2=\left(\frac{g_2}{4}(\delta-1)^2(\delta+1)\right)^{-1/(\delta-1)}. \tag{9}$$

We proceed in a similar way as for the one-dimensional case, considering the partition function $Z_2=\int\mathcal{D}[\phi]e^{-\Gamma_c[\phi]}$. In the path summation contribute, similarly to the one-dimensional (1D) case, saddle points for which $\vec{r}_o$ lies outside the cluster C. The main effect in the statistical mechanics of the 2D system is obtained through the summation over paths with $|\vec{r}_o|\geqslant R(R$ is, once again, the radius of $C)$ which are in fact constant configurations determined by the 2D parameter $\vec{r}_o$. In close analogy with the one-dimensional treatment, we write the path integral in Z_2 as an ordinary integral over $\vec{r}_o$. In this regime (constant configurations) the two-dimensional effective action is

$$\Gamma_c=G_2R^2r_o^{-2[(\delta+1)/(\delta-1)]},$$

with

$$G_2=\pi g_1\left[\frac{2A_2^2}{(\delta-1)^2}+g_2A_2^{\delta+1}\right]=2\pi g_1g_2A_2^{\delta+1}\left(\frac{\delta+3}{4}\right).$$

Performing the calculation of the mean value of the magnetization $\langle M(R)\rangle=\langle\int d^2\vec{r}\phi(\vec{r})\rangle$, characterizing the two-dimensional critical cluster and using the notation $R_d=A_2^{-(\delta+1)/2\delta}\mu^{(\delta+1)/2\delta}G_2^{1/2\delta}$ and $R_u=G_2^{(\delta-1)/4}$, we find the following

(i) For $R_d\ll R\ll R_u$,

$$\langle M(R)\rangle\sim R^{2\delta/(\delta+1)}, \tag{10}$$

with coefficient

$$\alpha_2\approx\pi A_2G_2^{-1/(\delta+1)}\frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)}.$$

This suggests the formation of a geometrical stucture in C with fractal mass dimension $d_F=2\delta/(\delta+1)$.

(ii) This behavior crosses over for $R\gg R_u$ to a power law $\langle M(R)\rangle\sim R^{2[(\delta-2)/(\delta-1)]}$ describing a local fractal with mass dimension $\tilde{d}_F=2(\delta-2)/(\delta-1)$ at large scales.

(iii) Finally, as in the 1D case, for $R\ll R_d$ the fractality is broken and the mass dimension coincides with the embedding dimension.

The extension to dimensions $d\geqslant3$ needs more care. In this case we must take into account the relation between the isothermal exponent δ and the anomalous dimension η: $\delta=(d+2-\eta)/(d-2+\eta)$ [9]. But let us first examine the case $\eta=0$. Repeating the procedure followed in the 1D and 2D cases we analytically obtain the saddle points of the d-dimensional critical effective action:

$$\phi=A_d(r_o^2-r^2)^{(2-d)/2}, \quad A_d=\left[\frac{d-2}{(2g_2)1/2}\right]^{(d-2)/2}r_o^{(d-2)/2}, \tag{11}$$

and transforming the path summation in the partition function Z_d into an ordinary integration over r_o, we find

$$\langle M(R)\rangle=<\int_c d^dr\,\phi>\sim R^{1+d/2} \tag{12}$$

for $R_d\ll R<\infty$. This means that, for $d=3$, $R_u\to\infty$. That is, the crossover to the second fractal has disappeared. What happens now if we take $\eta\neq0$ into account? Consider the case $0<\eta<1$. For a wide range of universality classes, including the $O(4)$ theory where $\eta\approx0.034$ [10], the anomalous dimension η obeys this condition. Actually for 3D systems η is very close to zero [11]. The corresponding Euler-Lagrange equation

$$\nabla_d^2\phi=(\delta+1)g_2\phi^{(d+2-\eta)/(d-2+\eta)} \tag{13}$$

cannot be solved exactly. Only an approximate instanton-like solution can be obtained analytically:

$$\phi_d(r)=A_d(r_o^2-r^2)^{(2-d)/2},$$
$$A_d=\left(\frac{(d-2)r_o}{\sqrt{2g_2}}\right)^{(d-2)/2}\left(\frac{d-2}{\sqrt{2g_2}r_o}\right)^{d\eta/4}. \tag{14}$$

Details concerning this calculation are given in the Appendix. Based on solution (14), and following the process applied for one and two dimensions, we determine $\langle M(R)\rangle$ for spherical symmetric clusters in $d\geqslant3$ dimensions. Using $R_d=a^{-(\delta+1)/d\delta}\mu^{(\delta+1)/d\delta}G_d^{1/d\delta}$ and $R_u=G_d^{-1/[d+q(\delta+1)]}$ with

$$a=\left(\frac{d-2}{\sqrt{2g_2}}\right)^{[d-2+(d\eta/2)]/2},$$

$$G_d=\frac{2a^{\delta+1}\pi^d/2}{d\Gamma(d/2)}g_1g_2,$$

and

$$q=\frac{2-d-\frac{d\eta}{2}}{2},$$

we obtain the following.

(i) For $R_d\ll R\ll R_u$,

$$\langle M(R)\rangle\sim R^{1+[(d-\eta)/2]} \tag{15}$$

with coefficient

$$\alpha_d\approx\frac{2\pi^{d/2}}{d\Gamma(d/2)}G_d^{-1/(\delta+1)}a\frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)}.$$

(ii) For $R\gg R_u$, the power law

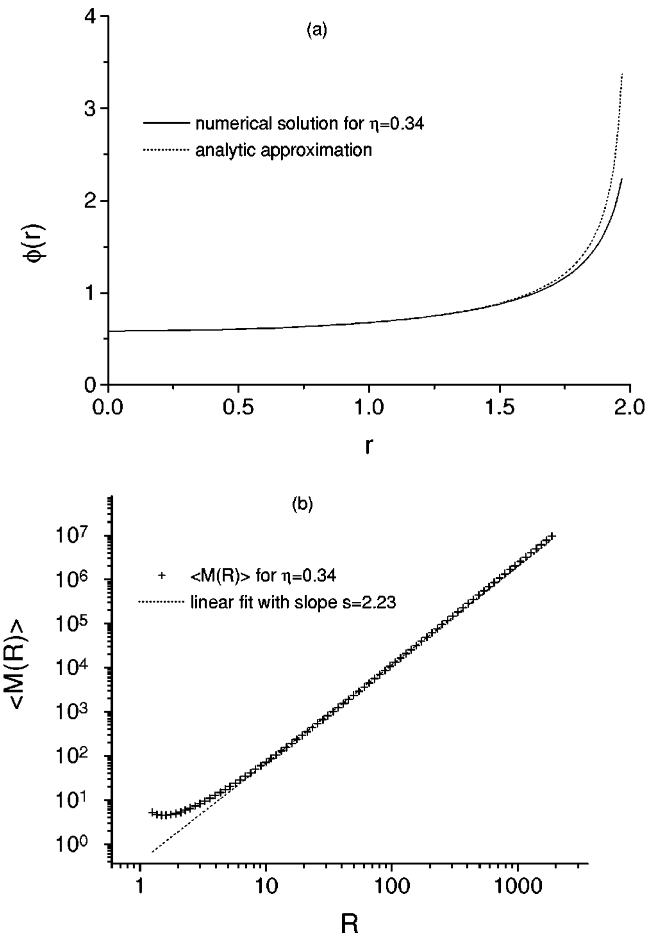

FIG. 2. (a) A typical 3D instanton-like saddle point for $\eta = 0.34$. Both the analytic approximation (dotted line) and the numerical calculation (solid line) are shown. (b) The mean magnetization $\langle M(R) \rangle$ for the 3D case using saddle points of the form presented in (a) to perform the corresponding statistical averaging. As in Fig. 1, the displayed quantities are in arbitrary units.

$$\langle M(R) \rangle \sim R^{1+[d(2-\eta)/4]}.$$

(iii) A breaking of the fractality for $R \ll R_d$.

Comparing Eqs. (12) and (15) we find, for $\eta = 0$, the same power law. This serves as a consistency check of the approximation we used. We calculated the saddle points of Eq. (13) numerically. We also calculated, in the constant configuration regime, the mean magnetization $\langle M(R) \rangle$. The results are presented in Fig. 2. In Fig. 2(a) we plot both the numerical and approximate solutions to the Euler-Lagrange equation for $d=3$ and $\eta = 0.34$.[1] The characteristic behavior of $\langle M(R) \rangle$ for $d=3$ is presented in Fig. 2(b). Here we have used $G_3 = 10^2$ and $\eta = 0.34$. The breaking of the fractality (for $R \ll R_d$) is clearly reproduced, while the crossover is suppressed due to the small value of η.

Putting together our results for one, two, and three dimensions, and denoting by d_F the fractal dimension in the central region of the cluster C and by $\tilde{d}_F$ the fractal dimension beyond the corresponding upper limit R_u, we have found

$$d_F = \frac{d\delta}{\delta+1}, \quad d=1,2,3,\ldots, \tag{16}$$

$$\tilde{d}_F = d - \frac{2}{\delta-1}, \quad d=1,2, \tag{17}$$

while for $d \geqslant 3$ we have

$$d_F - \tilde{d}_F = \frac{\eta(d-2)}{4} + O(\eta^2). \tag{18}$$

A remarkable property of the geometry of the cluster is that $d_F > \tilde{d}_F$ for all dimensions, indicating a dilution of the cluster C for distances greater than R_u from the center of the cluster. In other words, the size R_u of the cluster gives a measure of the correlation length in the finite system. For 3D systems, however, the maximal size of a single cluster (R_u) coincides practically with the size of the whole system ($\eta \approx 0$, $d_F \approx \tilde{d}_F$), and one recovers the conventional behavior of the correlation length ξ in a second-order phase transition (ξ is of the order of magnitude of the size of the system). For critical systems of low dimensionality ($d<3$) the association of the correlation length with the size of the system needs particular care, and this issue will be discussed in detail in Sec. IV.

IV. EXTENSIONS AND FINITE-SIZE EFFECTS

In Sec. III we showed the appearance of a fractal geometry for cluster C in the central region of scales $R_d \ll R \ll R_u$. For $R < R_d$ we obtain the breaking of fractality, and beyond R_u a more dilute fractal emerges. Therefore the limits R_u and R_d determine the part of the cluster with fractal dimension d_F. In the following we will investigate how a change of the coarse graining scale Λ affects the fractality region.

We consider the transformation

$$\Lambda^{-1} = \lambda \Lambda'^{-1}, \tag{19}$$

where the ultraviolet cutoff Λ fixes the coarse-graining scale (Λ^{-1}). Then Eq. (2) becomes

$$\Gamma_c[\phi] = g_1 \lambda^{d+2} \Lambda'^{-(d+2)} \int d^d x \left[\frac{1}{2} (\nabla_d \phi)^2 + g_2 \lambda^{-(2d+2)} \Lambda'^{2d+2} |\lambda^d \Lambda'^{-d} \phi|^{\delta+1} \right]. \tag{20}$$

Setting $\hat{\phi}' = \Lambda'^{-d} \phi$ and $\hat{x}' = \Lambda' x$, Eq. (20) simplifies to

$$\Gamma_c[\phi] = g_1 \lambda^{d+2} \int d^d \hat{x}' \left[\frac{1}{2} (\nabla_d \hat{\phi}')^2 + g_2 \lambda^{d(\delta-2)+(d-2)} |\hat{\phi}'|^{\delta+1} \right],$$

where the new constants g_1' and g_2' have the values

[1] We used this value instead of $\eta = 0.034$ valid for $O(4)$ in order to magnify the difference between the approximate and numerical solutions.

$$g_1' = g_1 \lambda^{d+2},$$
$$g_2' = g_2 \lambda^{d(\delta-2)+(d-2)}. \tag{21}$$

We have seen that the dimensionless values of R_u and R_d for the 1D case are $R_u = G_1^{(\delta-1)/(\delta+3)} \sim g_1^{(\delta-1)/(\delta+3)} g_2^{-2/(\delta+3)}$ and $R_d = A_1^{-(\delta+1)/\delta} G_1^{1/\delta} \sim g_1^{1/\delta} g_2^{1/\delta}$. According to Eq. (21) the new values for the dimensionless limits are $R_u' = \lambda R_u$ and $R_d' = \lambda R_d$. Using Eq. (19), we find that the quantities $\Lambda^{-1} R_u$ and $\Lambda^{-1} R_d$ do not depend on the choice of the cutoff Λ:

$$\Lambda'^{-1} R_u' = \Lambda^{-1} R_u,$$
$$\Lambda'^{-1} R_d' = \Lambda^{-1} R_d. \tag{22}$$

We may now extend these calculations for the case $d=2$, where $R_u = G_2^{(\delta-1)/4} \sim g_1^{(\delta-1)/4} g_2^{-1/2}$ and $R_d = A_2^{-(\delta+1)/2\delta} G_2^{1/2\delta} \sim g_1^{1/2\delta} g_2^{1/2\delta}$, and we again obtain Eqs. (22). More complicated is the case $d \geqslant 3$. If we neglect the anomalous dimension η we recover Eqs. (22), but taking η into account we cannot find an analytic expression for the limits R_d and R_u. However, using the approximate solutions (14), we can prove the validity of Eqs. (22) to a leading order in η.

Let us now consider the case when the order parameter is not directly the scalar field $\phi(x)$ but a power $\phi^n(x)$, $n > 0$. The extensive variable characterizing the critical geometry is now taken to be

$$M(R) = \int_C \phi^n(\vec{x}) d^d x.$$

Performing the calculation of $\langle M(R) \rangle$ at the level of the saddle-point approximation, we obtain the following

(i) In the central fractality region $R_d^{(d)} \ll R \ll R_u^{(d)}$, the dimension is

$$d_F^{(d)} = d\left(1 - \frac{n}{\delta+1}\right), \tag{23}$$

where the embedding dimension d takes the values: $d = 1,2,3,\ldots$.

(ii) The geometry in the external region $R \gg R_u^{(d)}$ is described through the dimension

$$\tilde{d}_F^{(d)} = d - \frac{2n}{\delta-1}, \quad d = 1,2, \tag{24}$$

also valid for $d \geqslant 3$ if we neglect the anomalous dimension η.

Using the obvious condition $\tilde{d}_F^{(d)} \geqslant 0$, Eq. (24) leads to an upper limit for the value of n:

$$n \leqslant \frac{d(\delta-1)}{2}.$$

Using as an example the parameter values $d=1$ and $\delta=5$, we obtain $n=1$ and 2 as possible values of the power n. For this special choice of d and δ we obtain a remarkable property when the order parameter is ϕ^2: in this case $\tilde{d}_F^{(1)} = 0$, and therefore $R_u^{(1)}$ coincides with the correlation length. Concerning the limits $R_d^{(d)}$ and $R_u^{(d)}$ and their dependence on the power n, we find that (i) the upper limit $R_u^{(d)}$ does not depend on n, and (ii) the lower limit $R_u^{(d)}$ has the following form for a general n:

$$R_d^{(d)} = G_1^{n/[d(\delta+1-n)]} \left(\frac{A_d}{\mu^{1/n}} \right)^{-[n(\delta+1)]/[d(\delta+1-n)]}.$$

We now turn to the question of critical cluster formation in a finite system. The effective action [Eq. (2)] is in fact valid for the ideal case of an infinite system. In order to take finite-size effects into account in a consistent way we have to add the term $\frac{1}{2} m^2 \phi^2$ in Eq. (2). In the following we will consider the statistical mechanics, within the theoretical framework developed so far, of the modifed effective action, which includes the above mentioned quadratic (mass) term in ϕ. Thereby we restrict ourselves in the simplified 1D case, although our results can be extended to higher dimensions without difficulty. For the isothermal critical exponent we also use the value $\delta=5$. The central interest in our investigations is to determine the changes induced to the upper limit R_u of the central fractality region, due to the presence of the mass term. This may lead us to a better understanding of the relation between R_u and the correlation length of the critical system.

The saddle points of the modified action are obtained (in the 1D case) using the energy integral [Eq. (5)]. The dominant configurations are those for which $E=0$. We then have

$$\int_0^x d\xi = \pm \int_{\phi(0)}^{\phi(x)} \frac{d\phi}{(m^2\phi^2 + 2g_2\phi^6)^{1/2}}. \tag{25}$$

Solving Eq. (25) for ϕ, we obtain

$$\phi(x) = \pm \frac{\sqrt{2\tilde{c}}\, m e^{\pm mx}}{(m^2 - 2g_2 \tilde{c}^2 e^{\pm 4mx})^{1/2}},$$
$$\tilde{c} = \frac{\phi^2(0)}{1 + \sqrt{1 + \dfrac{2g_2}{m^2}\phi^4(0)}}. \tag{26}$$

Solution (26) for small m simplifies to

$$\phi(x) = \sqrt{\frac{m}{4g_2\tilde{c}}} |x - x_o|^{-1/2}, \quad x_o = \frac{m^2 - 2g_2\tilde{c}^2}{8g_2\tilde{c}^2 m}, \tag{27}$$

and taking the limit $m \to 0$ we recover Eq. (6) for $\delta=5$.

The position of the singularity in Eq. (26) is

$$\tilde{x}_o = \pm \frac{1}{4m} \ln \frac{m^2}{2g_2\tilde{c}^2}. \tag{28}$$

It is easy to show that $\tilde{x}_o \to x_o$ [see Eq. (6)] for $m \to 0$. In the following and up to the end of this section we will, for simplicity, drop the tilde over x_o. Whenever x_o appears in the following formulas it means expression (28).

Inserting Eq. (28) into solution (26), we finally obtain

$$\phi(x)=\left(\sqrt{\frac{2}{g_2}}m\right)^{1/2}\frac{e^{m(x-x_o)}}{(1-e^{4m(x-x_o)})^{1/2}}. \tag{29}$$

Repeating the arguments of Sec. II we perform the path summation in the partition function of the finite system using the constant solutions, deduced from Eq. (29), for $x_o \gg x$:

$$\phi(x)=\left(\sqrt{\frac{2}{g_2}}m\right)^{1/2}\frac{e^{-mx_o}}{(1-e^{-4mx_o})^{1/2}}. \tag{30}$$

Within this approximation the effective action of the finite system is

$$\Gamma_c(R,x_o)=\tilde{G}R\frac{e^{-6mx_o}}{(1-e^{-4mx_o})^3}, \quad \tilde{G}=2^{5/2}g_1g_2^{-1/2}m^3. \tag{31}$$

The path summation in the partition function goes over to an ordinary integration over x_o, with a measure obtained from Eq. (30):

$$Z\sim\int_R^{x_{o_{max}}}dx_o[m^{3/2}e^{-mx_o}(1-e^{-4mx_o})^{-1/2} - m^{3/2}e^{-5mx_0}(1-e^{-4mx_0})^{-3/2}] \times\exp\left(-\tilde{G}R\frac{e^{-6mx_o}}{(1-e^{-4mx_o})^3}\right). \tag{32}$$

In the limit $m\rightarrow 0$, Eq. (32) becomes

$$Z\sim\int_R^{x_{omax}}dx_o x_o^{-3/2}e^{-G_1Rx_o^{-3}},$$

recovering the expression for the partition function Z_1 found in Sec. II.

Now setting

$$\omega(z)=\frac{\tilde{G}R}{8\sinh^3(2mz)},$$

from Eq. (32) we obtain

$$Z\sim\int_{\omega(x_{o,max})}^{\omega(R)}dt f(t)e^{-t}, \tag{33}$$

where $f(t)$ is given by

$$f(t)=m^{1/2}\left\{\left(\frac{\tilde{G}R}{t}\right)^{1/6}-\left[\sqrt{1+\frac{1}{4}\left(\frac{\tilde{G}R}{t}\right)^{2/3}}-\frac{1}{2}\left(\frac{\tilde{G}R}{t}\right)^{1/3}\right]\times\left(\frac{\tilde{G}R}{t}\right)^{1/2}\right\}\frac{(\tilde{G}R)^{-1}}{\sqrt{1+\frac{1}{4}\left(\frac{\tilde{G}R}{t}\right)^{2/3}}}\left(\frac{\tilde{G}R}{t}\right)^{1/3}. \tag{34}$$

The value of R_u is then determined through the condition $\omega(R_u)=1$. In the limit $m\rightarrow 0$ we find $R_u=2^{-7/4}g_1^{1/2}g_2^{-1/4}$, in accordance with the results obtained in Sec. II. For $m\gg 1$, we obtain an analytic expression for R_u:

$$R_u=\frac{ln m}{2m}. \tag{35}$$

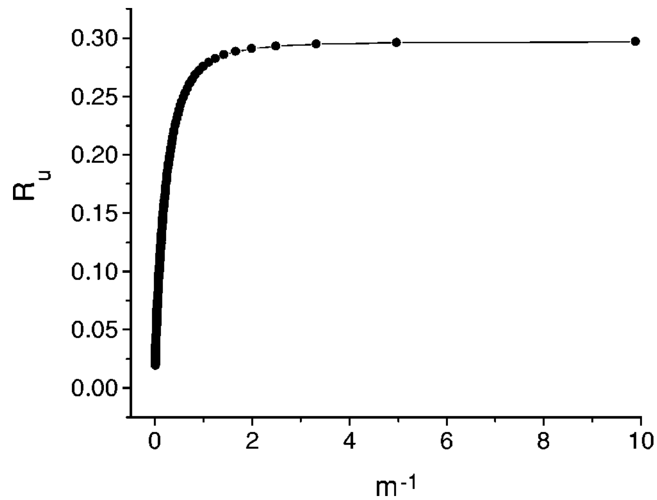

FIG. 3. The upper limit R_u of the central fractality region for $d=1$ as a function of the size $1/m$ of the critical system. We use arbitrary units for R_u and m.

For general m, however, R_u can be determined only numerically. In Fig. 3 we present the results for the dependence of R_u on m, numerically solving the equation $\omega(R_u)=1$. We have used $\tilde{G}=1$. The quantity $1/m$ is actually the linear size of the critical system. For a large system ($m^{-1}\gg 1$), the cluster size R_u becomes saturated, becoming independent of m. The long range correlation, in this case, is generated by succesive convolutions of neighboring clusters. In other words, the picture for the global system, emerging from our results (Fig. 3), is a superposition of fractal clusters with finite size, which, by coalescense, may create long range ordering in the critical system. In order to illustrate this effect, we have constructed, by simulation, a global system in two dimensions as a set of softly interacting clusters with prescribed geometrical properties. In fact, based on a mean field approximation scheme, it is straightforward to determine the distribution of N clusters, $P(\vec{R}_1,\ldots,\vec{R}_N)$, with centers located at the points $(\vec{R}_1,\ldots,\vec{R}_N)$ in the area S_g of the global system. For this purpose we consider the potential term in the effective action,

$$U(\overline{\phi})=g_1g_2\int_{S_r}d^2\vec{r}\,'[\overline{\phi}(\vec{r}\,')]^{\delta+1}, \qquad \overline{\phi}=\frac{1}{S_r}\left\langle\int_{S_r}d^2\vec{r}\,'\phi(\vec{r}\,')\right\rangle, \tag{36}$$

where S_r is the area occupied by a cluster of radius r, and the mean field $\overline{\phi}(\vec{r})$ is written, according to our results in Sec. III, as follows:

$$\overline{\phi}(\vec{r}) = \frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)}\left(2\pi g_1 g_2 \frac{(\delta+3)}{4}\right)^{-1/(\delta+1)} r^{-2/(\delta+1)}$$

$$(R_d \leqslant r \leqslant R_u) \tag{37}$$

From Eqs. (36) and (37), we obtain

$$U(\overline{\phi}) = \frac{1}{2\pi}\left(\frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)}\right)^{\delta+1}\left(\frac{4}{\delta+3}\right)\int_0^{2\pi} d\theta \ln\left(\frac{R(\theta)}{R_d}\right), \tag{38}$$

where $R(\theta)$ specifies the distortion of the area occupied by the cluster in question, owing to a state of coalescense with neighboring clusters. Introducing the mean radius $\overline{R} = R(\overline{\theta})$ in integral (38), we finally obtain

$$U(\overline{\phi}) = \frac{1}{2}\left(\frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)}\right)^{\delta+1}\left(\frac{4}{\delta+3}\right)\ln\left(\frac{\overline{S}}{S_d}\right),$$

$$S_d = \pi R_d^2, \quad \overline{S} = \pi \overline{R}^2, \tag{39}$$

where $\overline{S}$ is a measure of the area occupied by a distorted (in general) cluster. The distribution of N clusters, in this picture, $P(\vec{R}_1, \ldots, \vec{R}_N) \sim \Pi_{i=1}^{N} e^{-U_i(\overline{\phi})}$, is given by the following area law:

$$P(\vec{R}_1, \ldots, \vec{R}_N) = Z_N^{-1} S_d^{N\alpha_\delta}(\overline{S}_1 \overline{S}_2, \ldots, \overline{S}_N)^{-\alpha_\delta},$$

$$\alpha_\delta = \frac{1}{2}\left(\frac{\Gamma\left(\frac{2}{\delta+1}\right)}{\Gamma\left(\frac{1}{\delta+1}\right)}\right)^{\delta+1}\left(\frac{4}{\delta+3}\right), \tag{40}$$

$$Z_N = \frac{S_d^{N\alpha_\delta}}{N!}\int_{S_g} d^2\vec{R}_1, \ldots, d^2\vec{R}_N (\overline{S}_1 \overline{S}_2, \ldots, \overline{S}_N)^{-\alpha_\delta}.$$

The smallness of the exponent α_δ in Eqs. (40) ($\alpha_\delta \approx 0.006$ for $\delta = 5$) guarantees that the interaction of clusters is very weak, leading to a random distribution $P(\vec{R}_1, \ldots, \vec{R}_N)$ in the area S_g and to a Poisson behavior of the partition function, $Z_N \approx S_d^{N\alpha_\delta}(S_g^N/N!)$. It permits us, therefore, to treat the global system as an almost ideal gas of clusters. The parameters of the effective theory, as well as the size of the critical system, determine then the number of formed clusters, the density within each cluster, and the size of each cluster.

It is now straightforward to construct the global system as a set of softly interacting clusters with prescribed geometrical properties. For simplicity let us consider a system for which the order parameter is interpreted as density of particles. Given the linear size R, the isothermal critical exponent δ and the effective couplings g_1 and g_2 we calculate the size R_u of a single cluster and the corresponding number of clusters $N_{cl} = (R/R_u)^2$ in the system. The number of particles n within each cluster is then given through Eq. (10). The entire net of clusters is constructed in two steps. First we generate the positions of the centers of the clusters, treating them as uniformly distributed random variables over a square with site R. Then we generate the points inside each cluster with a distribution function specified by Eq. (10). If two clusters (say the ith and the jth clusters) overlap, then a point in the ith cluster is taken into account if the nearest center to it is the center of the ith cluster, otherwise this point is neglected. Such a construction is presented in Fig. 4. We have used the parameter values $R = 1$, $\delta = 5$, $g_1 = 50$, and $g_2 = 1$.

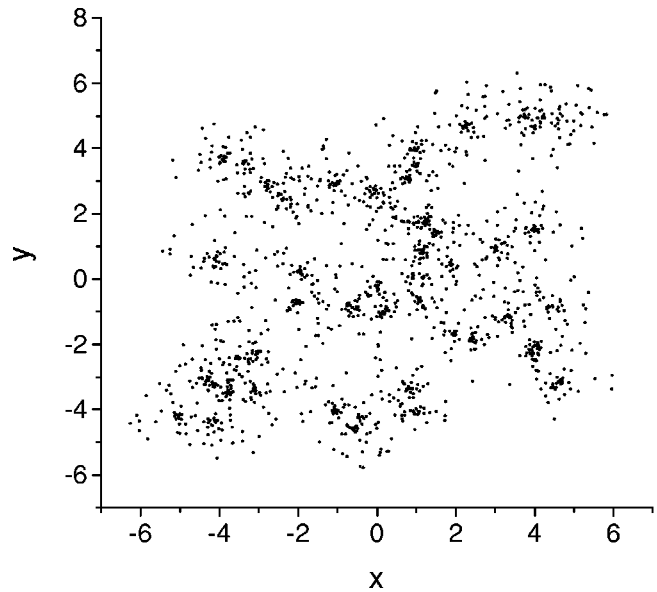

FIG. 4. The global 2D critical system described through an effective action of the form of Eq. (1). We used the parameters $g_1 = 50$, $g_2 = 1$, and $\delta = 5$. Each full circle represents a point $\vec{x}$ in the 2D space occupied by the critical system with $\phi(\vec{x}) > \phi_{min}$. The coordinates x and y are given in arbitrary units.

Although the fractal mass dimension of each cluster is the same, the resulting global set turns out to have a different fractal dimension. In fact, calculating the generalized dimensions of the global set we observe that its geometric structure does not correspond to a pure monofractal set. Here we have used the method of factorial moments to perform this analysis [12]. We divide the region of the global system into M^2 boxes of linear size $l(M = R/l)$. Denoting by n_i the number of points within the ith box we define the pth order factorial moment of the distribution of the points of the global set in space as

$$F^{(p)}(M) = \frac{\frac{1}{M^2}\sum_{i=1}^{M^2} n_i(n_i - 1)\cdots(n_i - p + 1)}{\left(\frac{1}{M^2}\sum_{i=1}^{M^2} n_i\right)^p}. \tag{41}$$

For a fixed value of p, the moment $F^{(p)}(M)$ possesses a power-law dependence on M (for $M \gg 1$), provided that the

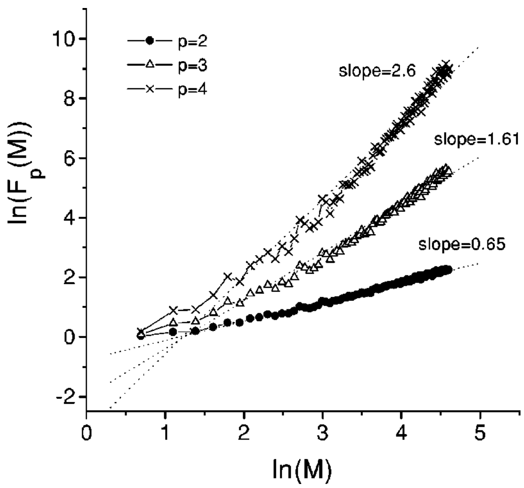

FIG. 5. The first three factorial moments for the point set presented in Fig. 4. The linear fits indicate the slopes of the corresponding moments suggesting the multifractality of the underlying set.

point-set under consideration has a fractal structure: $F^{(p)}(M) \sim M^{s_p}$. The exponent s_p is related to the fractal dimension of the corresponding point set: $s_p = (p-1)(d - d_p)$. The dimensions d_p are the generalized dimensions characterizing the system d_2, being the corresponding average fractal dimension. For a monofractal set the generalized dimension spectrum is given as $d_p = d_2$ for $p = 3,4, \ldots$. We have calculated the first three factorial moments ($p = 2$, 3, and 4) as a function of M for the set shown in Fig. 4. The results are shown in a log-log plot in Fig. 5. We find the exponents (slopes in the log-log plot) $s_2 = 0.65$, $s_3 = 1.61$, and $s_4 = 2.6$, suggesting that the underlying set is a multifractal. A deeper understanding of the dimension spectrum of the global system based on the construction described above is a subject for a future investigation.

V. CONCLUSIONS

We have studied in detail the formation of critical clusters in a wide class of systems, undergoing a thermal phase transition of second order. We have used suitable, instanton-like, saddle-point configurations for the local-field fluctuations, in order to saturate the path summation of the partition function. In this treatment we were able to describe the critical system locally, and specify the geometrical properties of a single critical cluster in terms of the parameters of the effective action, at the critical point. Our main results are summarized as follows.

(1) In critical systems of low dimensionality ($d < 3$), there exist two characteristic scales (R_d, R_u) which specify the geometry of any single critical cluster, namely, its maximal size (R_u) and its fractality region $R_d \ll R \ll R_u$. The fractal dimension is $d_F = \delta d/(\delta + 1)$, in agreement with other treatments on a lattice, and the minimal scale R_d, below which fractality breaks down, is the analog of the lattice spacing in any treatment in discrete space.

(2) Beyond the scale R_u ($R \gg R_u$), the fractal dimension crosses over to smaller values, $\tilde{d}_F < d_F$, and for a suitable choice of the order parameter [$\phi^n(x)$ with $n = d(\delta - 1)/2$] it may even vanish ($\tilde{d}_F = 0$). Therefore, the scale R_u can be associated with the direct correlation length, $\xi_d \approx R_u \Lambda^{-1}$ [9], which coincides with the maximal size of a single cluster.

(3) For sufficiently large systems (size $\gg \Lambda^{-1}$) the direct correlation length ($\xi_d \approx R_u \Lambda^{-1}$) is independent of the size of the system (Fig. 3), and remains finite even in the thermodynamic limit (infinite system).

(4) The global system is built up by a random distribution of critical clusters which may overlap, giving rise to a long-range total correlation, and therefore to density fluctuations at all scales. We have shown the validity of such a mechanism in two dimensions by developing a suitable algorithm in order to construct the global system, based on the local description. Our results show that the entire critical system develops strong density fluctuations, of multifractal nature, in a wide range of scales, far beyond the size of a single cluster (Figs. 4 and 5). A deeper understanding of this global structure, and in particular of the connection between the fractal geometry of a single cluster and the multifractal spectrum of the entire system, remains a challenging, open question.

(5) In 3D critical systems, the anomalous dimension is approximately zero ($\eta \approx 0$), and the crossover scale R_u associated with a single cluster coincides practically with the size of the global system ($d_F \approx \tilde{d}_F$). This observation leads to a simple picture according to which the development of fluctuations at all scales, at the critical point, is realized through the formation of self-similar clusters at all sizes. The maximum cluster size (R_u) coincides with the size of the global system, and gives a measure of the correlation length which becomes infinitely large, in the thermodynamic limit (infinite system). In this case the geometry of the global system coincides with the geometry of a single cluster and therefore it remains monofractal with the same fractal dimension d_F.

Concluding, we have investigated the geometrical structure of critical fluctuations, developed locally in a thermal system which is described effectively by a scalar field. The formation of fractal clusters with a mass dimension d_F, fixed by the universality class, has been revealed, and the mechanism for generating fluctuations at all scales, in the entire system, based on the local properties, has been discussed. It is of interest to note that the fluctuations of the global system obey a different geometry in 3D critical systems (monofractal), and in critical systems of low dimensionality, $d < 3$ (multifractal). The reason for this different behavior is due essentially to the fact that in 3D systems, described by scalar theories, the anomalous dimension η turns out to be very small ($\eta \approx 0$).

APPENDIX

Integrating the first-order differential equation (5), we obtain

$$\int_0^x d\xi = \pm \int_{\phi(0)}^{\phi(x)} \frac{d\phi}{[2(E + g_2 \phi^{\delta+1})]^{1/2}}.$$

It follows that

$$x=\pm\frac{1}{\sqrt{2E}}\left(\frac{E}{g_2}\right)^{1/a}I, \quad \text{(A1)}$$

where

$$\begin{aligned} a&=\delta+1,\\ I&=\int_{r_1^a}^{r_2^a}\frac{\psi^{\frac{1}{a}-1}}{(1+\psi)^{1/2}}d\psi,\\ \psi&=\frac{g_2}{E}\phi^a,\\ r_1&=\left(\frac{g_2}{E}\right)^{1/a}\phi(0),\\ r_2&=\left(\frac{g_2}{E}\right)^{1/a}\phi(x). \end{aligned} \quad \text{(A2)}$$

The integral I can be determined analytically:

$$\begin{aligned} I&=\int_0^{r_2^a}\frac{\psi^{(1/a)-1}}{(1+\psi)^{1/2}}d\psi-\int_0^{r_1^a}\frac{\psi^{(1/a)-1}}{(1+\psi)^{1/2}}d\psi\\ &=a\left[r_2\,{}_2F_1\left(1/2,1/a;1+\frac{1}{a},-r_2^a\right)\right.\\ &\quad\left.-r_1\,{}_2F_1\left(1/2,1/a;1+\frac{1}{a},-r_1^a\right)\right], \end{aligned} \quad \text{(A3)}$$

where ${}_2F_1$ is the hypergeometric function.

Inserting the formula [13]

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z)&=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-1)^{\alpha}z^{-\alpha}{}_2F_1\\ &\quad\times\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}\right)\\ &\quad+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-1)^{\beta}z^{-\beta}{}_2F_1\\ &\quad\times\left(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{z}\right) \end{aligned}$$

into Eq. (A3) we obtain

$$x=\pm\frac{c_1}{\sqrt{2g_2}}(\phi(x)^{1-(1/a)}f_1-\phi(0)^{1-(1/a)}f_2), \quad \text{(A4)}$$

with

$$c_1=\frac{\Gamma\left(1+\frac{1}{a}\right)\Gamma\left(\frac{1}{a}-\frac{1}{2}\right)}{\Gamma(1/a)\Gamma\left(\frac{1}{2}+\frac{1}{a}\right)}=-\frac{2}{\delta-1} \quad \text{(A5)}$$

and

$$\begin{aligned} f_1&={}_2F_1\left(1/2,\frac{1}{2}-\frac{1}{a},\frac{3}{2}-\frac{1}{a},-\frac{E}{g_2}\phi(x)^{-a}\right),\\ f_2&={}_2F_1\left(1/2,\frac{1}{2}-\frac{1}{a},\frac{3}{2}-\frac{1}{a},-\frac{E}{g_2}\phi(0)^{-a}\right). \end{aligned} \quad \text{(A6)}$$

Then Eq. (A4), for $E\to 0$, leads to solution (6),

$$\phi(x)=\left[\frac{(\delta-1)^2g_2}{2}\right]^{-1/(\delta-1)}|x_o-x|^{-2/(\delta-1)},$$

with

$$x_o=x_o(\phi(0))=\frac{2}{(\delta-1)\sqrt{2g_2}}(\phi(0))^{(1-\delta)/2}. \quad \text{(A7)}$$

From Eq. (A4), and for $E\neq 0$, we obtain that

$$\phi(x)=\left[\frac{(\delta-1)^2g_2}{2}\right]^{-1/(\delta-1)}(x_o'-x)^{-2/(\delta-1)}\left(\frac{1}{f_1}\right)^{-2/(\delta-1)}, \quad \text{(A8)}$$

where now $x_o'=x_of_2(E)$. That is, if $E\neq 0$, x_o depends on two parameters:

$$x_o'=x_o'(\phi(0),E). \quad \text{(A9)}$$

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